Essay Review: Medieval Italian Practical Mathematics[1]

Jacopo da Firenze's Tractatus Algorismi *and Early Italian Abbacus Culture*, by Jens Høyrup (Basel: Birkhäuser, 2007), 482 pp. ISBN: 978-7643-8390-9. US$139.

1. Introduction to abbacus mathematics

The Italian vernacular texts on practical mathematics, flanked in time by Fibonacci's *Liber Abaci* (1228) and Luca Pacioli's *Summa de Arithmetica* (1494), have traditionally elicited little interest from historians. These abbacus[2] books appeared for the most part to be abridgements of Fibonacci's opus, with no interesting developments to speak of. Fortunately this picture has changed, thanks largely to the editions and commentaries by Gino Arrighi and his colleagues in Siena, and to Warren van Egmond's 1980 catalog of abbacus manuscripts. For the history of algebra, these *libri d'abbaco* show that the search for solutions to cubic and quartic equations in Christian Europe did not begin with Pacioli, but extends back at least to the first part of the 14th c.[3]

Abbacus books were ostensibly written in connection with the abbacus schools of central and northern Italy, where for two years or so the children of merchants and the nobility learned basic mathematics. The books contain instruction in arithmetic with Hindu-Arabic numerals, usually accompanied with worked-out problems (*ragioni*). Many of these problems are posed as mercantile questions, and they are solved by a number of methods, including the rule of three, false position, and algebra. Some books also cover mensuration.

The production of abbacus books extended from the late 13th c. to roughly the early 16th c. While the dating of manuscripts can often be determined by watermarks, tracing textual influences is a treacherous affair. The "authors" of these books frequently copied entire chapters from other books, and even when they did write on their own they often lifted problems and examples from other sources. In many cases the "writing" of an abbacus text is more like an adaptation of previous material to suit an author's needs. Van Egmond adds, "The great majority of the manuscript abbaci are anonymous, and for those few that do provide names, it is often unclear whether they are naming the original author, the copyist, or a respected authority."[4] Even for books faithfully reproduced, the copyist would often rewrite the book in his own dialect. For instance, two manuscripts of the *Aliabraa Argibra* of Maestro Dardi (1344) are written in Venetian (from ca. 1395) and Tuscan (ca. 1470) (p. 169, n. 326).

Previously the only abbacus book accessible to English readers was D. E. Smith's translation of *Arte del Abbacho* (*Art of the Abbacus*, printed in Treviso in 1478), which was published in 1987 in Frank Swetz's *Capitalism & Arithmetic: The New Math of the 15th Century*.[5] Translations of a couple

[1] Translations from Vat. Lat. 4826 are from Høyrup's book. All other translations are mine.
[2] The Italian word *abbaco/abaco* refers to calculation, and not to the comparatively recent eastern calculating instrument.
[3] In the Islamic world such investigations began no later than the 10th c. (Sinān ibn al-Fatḥ, ʿAlī al-Sulamī), but they do not seem to have been transmitted to Europe.
[4] [Van Egmond 1980, 27].
[5] [Swetz 1987].

chapters on algebra from other books have also appeared.[6] Jens Høyrup (hereafter H.) now gives us an edition and translation of an early treatise: the 1307 *Tractatus Algorismi* (*Algorithm Treatise*), composed in Montpellier by one Jacopo da Firenze. H.'s book is divided into two parts, roughly equal in size. The first part, "Jacopo, his treatise, and abbacus culture", is H.'s commentary. The second part, "The Vatican manuscript edition and translation", is followed with an appendix, "The revised version, Milan and Florence", containing a combined edition of the other two extant manuscripts.

Of the three manuscripts, only the Vatican example contains a treatment of algebra. Historians have previously held that these chapters were added later by a copyist. H. now tells us that they were part of Jacopo's original book, which is no small claim. If true, then Jacopo's algebra would predate by 21 years the next oldest Italian algebra, from Paolo Gherardi's 1328 *Libro di Ragioni* (*Book of Problems*).[7] Further, H. proposes that abbacus books are not abridgements of Fibonacci's *Liber Abaci*, but rather originated through a different channel of transmission from the Islamic world.

In this review I give an overview of the contents of Jacopo's book, with some comments on the translation. After that I analyze H.'s commentary. Briefly, I find that H.'s attribution of the chapter on algebra to Jacopo is fraught with difficulties. But he is correct that the abbacus tradition did not derive its material primarily from Fibonacci or other Latin works, though much of his evidence is invalid.

2. Jacopo da Firenze: three manuscripts and their contents

Jacopo's incipit tells us that he wrote the *Tractatus Algorismi* in 1307 in Montpellier. Three manuscripts are known, which H. labels **F**, **M**, and **V**:

F = Florence, Biblioteca Riccardiana 2236,

M = Milan, Trivulziana 90, dated by watermarks to ca. 1410, and

V = Vatican, Biblioteca Apostolica Vaticana Lat. 4826, dated by watermarks to ca. 1450.

H.'s edition and translation are based on the Vatican manuscript, which he divides into 22 chapters (p. 10). Of these, Chapters 16 to 19 and 22 are not found in **F** or **M**. Chapters 1 through 10 explain basic arithmetic: how to operate with whole numbers and fractions expressed with Arabic numerals, including tables of multiplication and division. Chapters 11 to 15 cover the rule of three, interest calculations, problems involving units of measurement, and mercantile, "recreational", and geometry problems.

Chapters 16 to 19 explain algebra and quasi-algebraic procedures. Chapter 16 gives the rules for solving the six simplified equations of degrees 1 and 2 together with several worked-out problems.[8] Chapter 17 gives the rules for reducible third and fourth degree equations, but without sample problems. H. puts the next problem, about mixing grains, in Chapter 18 by itself, since it does not fit with the contents around it. Chapter 19 contains problems whose solutions seem to be generalized from the rules given in Chapter 16. Chapter 20 gives a table of the fineness of coins, and Chapter 21 consists of alloying problems. Last is Chapter 22, offering more mixed problems of the same kind as Chapters 14 and 15.

[6] [Van Egmond 1978; Heeffer 2008].

[7] [Gherardi 1987; Van Egmond 1978].

[8] In modern notation, the six equations are $ax = b$, $ax^2 = b$, $ax^2 = bx$, $ax^2 + bx = c$, $ax^2 + c = bx$, and $ax^2 = bx + c$.

3. H.'s translation

The pages of the edition/translation are divided into two columns. The Italian text is on the left, and the English translation on the right. This makes cross-referencing easy. "The English translation attempts to keep very close to the text, and to render always in the same way the same phrase or term when used in the same function, even in cases where this implies some awkwardness." (p. 190). In this way H. succeeds in conveying the linguistic, social, and pedagogical dressing that make the abbacus text what it is. His translation may take some getting used to, but it is worth the effort.

The verbs used to "take" a fraction of a number illustrate H.'s approach. He translates three Italian verbs with three English verbs: *prendere* ("take"), *pigliare* ("seize"), and *togliere* ("grasp"). So we find phrases in the translation such as "take $\frac{2}{3}$ of 12, which are 8" (p. 232), "And then seize $\frac{1}{3}$ of 12 which is 4" (p. 230), and "Grasp $\frac{1}{3}$ of 19 and $\frac{1}{5}$, which is 6 and $\frac{2}{5}$" (p. 261).

A particularly awkward example is the way rates are expressed. While Jacopo's "doi deñ per £" is nicely rendered as "two *denari* per *libra*" (p. 253), many rates have the definite article *la/el/lo* in place of *per*. One of several examples is "Un homo à 100 staia de grano che vale ß 20 lo s̃", which H. translates with the literal, but downright confusing, "A man has 100 *statia* of grain that is worth *soldi* 20 the *statio*" (p. 323).

We rarely catch H. in an outright translation error. One example I found is in 11.6, where he translates "Dobiamo multiplicare per la ragione" as "We shall multiply after the computation" instead of "We shall multiply by the rate" (p. 238).

H.'s translation of *ragione* is odd. While he recognizes in his commentary that it means "problem"[9] (pp. 42, 118 n. 253), in the translation he regularly writes it as "computation". So for example, he begins problem 12.11 (p. 245) with, "If some computation should be given to us about interest" instead of "If some problem should be given to us about interest". The words become tangled in 16.14, where "Et abbiamo facto ragione che, conpiuti i viaggi, si trovò 54…" is translated as "And we have computed that, when the voyages were completed, he found himself with 54…" (p. 315) But *conpiuti* is "computed", not "completed". The passage is better, and more literally, rendered as "And we have made a problem [i.e. the previous problem, 16.13] that, when the voyages were computed, he found himself with 54…"

One other oddity worth mentioning is H.'s repeated translation of *dimme* as "say me" instead of "tell me". For instance, in 10.2 (p. 230) he has "say me, how much is joined together $\frac{1}{2}$ and $\frac{1}{3}$" (i.e. calculate $\frac{1}{2} + \frac{1}{3}$).

4. H.'s commentary

Many readers would have benefitted from a chapter giving a general introduction to abbacus culture and books. The closest H. comes is in the introduction to the chapter "The Abbacus Tradition" (pp. 27-30), which is provided more as a backdrop to one of his arguments than as background information for the non-specialist. But H. does devote over a hundred pages to explanatory remarks in his chapter "The Contents of Jacopo's Treatise" (pp. 45-146). This chapter-by-chapter rundown guides the reader regarding the mathematical content of **V** , as well as in the history of particular problems and techniques. The rest of the commentary, and indeed much of the "hundred pages", is

[9] In other contexts in Jacopo's book *ragione* means "rate", "reason", and even "share".

directed to specialists. For a general overview of abbacus books in English I recommend Warren van Egmond's article, "Abbacus arithmetic", or the "General Introduction" to van Egmond's *Practical Mathematics in the Italian Renaissance*.[10]

Prior to H.'s work in Italian mathematics, historians were comfortable with the assessment of Louis Karpinsky and Warren van Egmond that the chapters on algebra in **V** were a later addition, and that Paolo Gherardi's 1328 *Libro di Ragioni* contains the oldest extant treatment of algebra in Italian. Further, historians were in agreement that by and large abbacus books are abridgements and adaptations of Fibonacci's massive *Liber Abaci*. In his commentary and in previously published articles[11] H. argues instead that the Chapters 16-19 and 22 in **V** were part of Jacopo's original 1307 treatise, and that early abbacus books are not abridgements of the *Liber Abaci*. He defends his two main claims with other sub-claims, which I group as follows:

(1) Chapters 16-19 and 22 of **V**, which include algebra, belong to Jacopo's original treatise.

(1b) Further, Jacopo's algebra was the *very first* in Italian, and he learned it not from the environment of Montpellier or Italy, but from an unknown "area ?".

(2) Abbacus mathematics does not derive (much) from Fibonacci.

(2b) Further, abbacus algebra does not derive from Latin works or their Arabic originals.

H. arrives at (1) by a flawed analysis of word choice and orthography, and by an application of circular reasoning. Of course this affects his claims in (1b). Claim (2), however, is argued well. And while claim (2b) is true, it is not new. Van Egmond, for instance, saw that early abbacus algebra is distinct from the available Latin works.[12] But more importantly, much of H.'s evidence for (2b) is wrong and shows that he does not understand certain fundamentals of medieval algebra.

5. Claim (1): The extra chapters in V belong to Jacopo's original treatise

Here is a brief outline of H.'s approach, which is argued in the chapter "Three Manuscripts". By a "close textual analysis" of **V**, **F**, and **M** he concludes that, "**V** is a quite faithful descendent of Jacopo's original (or at least of the common archetype for all three manuscripts), whereas the closely related **F** and **M** are the outcome of a process of rewriting and abridgement...". (p. 5-6) From there he asserts that the uniformity of **V** with respect to language, together with some promised evidence on algebra, imply that the extra chapters must have been written by Jacopo himself.

After spending the first part of the chapter describing the manuscripts, H. sets about to show that one of **V**, **F**, or **M** is a faithful copy of their common archetype. He starts by comparing the spellings *fact...* vs. *fatt...*[13] (from *facto*, "to do, to make") in **V** and **F**. Among the 35 places where both manuscripts have the word, there are 7 instances where both have the spelling *ct*, 8 where both have *tt*, and 20 where **V** has *ct* and **F** has *tt*. There is no instance where **V** has *tt* and **F** *ct*. H. writes, "If the spellings of both manuscripts had resulted from independent variation with respect to the archetype (the scribe of **V** mostly preferring *ct*, the one of **F** mostly *tt*), the 7 *fact* of **F** would have been distributed randomly over the relevant 35 *fact*+*fatt* of **V** (or, reciprocally, the 8 *fatt* of **V** randomly over the relevant 35

[10] [Van Egmond 1992; Van Egmond 1980, 3-33].
[11] The main articles are his [2000], [2001], [2005], and [2006], which are not listed in the book's bibliography.
[12] [Van Egmond 1978, 158; Van Egmond 2007].
[13] An example of *fact* is in the statement "Et è facta, et sta bene" ("And it is done, and it goes well", p. 367).

fact+*fatt* of **F**). In this situation the odds are around 13.2% that no *fatt* in **V** will correspond to *fact* in **F** – namely $\frac{28!\cdot 27!}{20!\cdot 35!}$." (p. 15 n. 33)

Because scribes *do* have preferences, the spellings will *not* result from random variation. The scribe responsible for **F** would not change a *tt* to a *ct*, nor would the scribe of **V** change a *ct* to a *tt*. All we can say is that the common archetype (probably) had the 7 *ct*s and the 8 *tt*s common to both manuscripts. We can say nothing about the original spellings of the 20 other instances of the word.

The same kind of invalid reasoning is then applied to the choices of the words *prendere* vs. *pigliare*, and *partire in* vs. *partire per*. From all this H. draws the conclusion that one of **V**, **F**, or **M** must be close to the original regarding these choices (p. 16). But it is easy to see that the common source of the three manuscripts may have been something in between them all, and the preferences of different scribes led to the versions we now have.

From here H. shows that **F** and **M** are the result of a reworking of the text, mainly by showing that they are not reliable regarding the mathematics. But without knowing that one of the three manuscripts is close to the original, this does nothing to advance his claim.

At the end of the chapter (pp. 23-25), H. appeals to the homogeneity of **V** to show that the extra Chapters 16-19 and 22 were written by Jacopo. "On all levels – orthography, vocabulary, discourse, pedagogical style – the treatise is a seamless whole…" (p. 24) Because H. failed to show that **V** is a faithful copy of Jacopo's original for the common chapters, it makes more sense that a later scribe realigned a hybrid text to his own dialect and word preferences. The only characteristic element of discourse or pedagogical style which link the extra chapters with the rest of the treatise is the use of the "partnership model", which alone is too weak to support his claim.[14]

In the last paragraph of the chapter H. admits that his arguments do not prove that the extra chapters are Jacopo's. "As we shall see, at least the algebra [of **V**] must be dated well before Paolo Gherardi's work from 1328, which leaves very little time for insertion of extra material into the original treatise and for a thorough reworking by an independent hand. All in all, the most reasonable assumption is thus that the algebra (and the supplementary mixed problems of Chapter 22) belong to the original treatise, and that **V** as a whole reproduces all major and most minor features of Jacopo's treatise faithfully." (p. 25). But looking ahead to the chapters which discuss algebra (pp. 100-115, 147-182), we do not find the promised evidence. Instead, he treats his conclusion as already established!

H. makes one other pitch to link the extra chapters with the common chapters. He notes that Chapter 22 "seems to overlap Chapters 14-15. At closer inspection, however, the apparent overlap turns out to consist of duly cross-referenced variations and supplements; no single genuine repetition can be found. This would hardly be the case if a later hand had glued another problem collection onto an original shorter treatise…" (p. 24) In Chapter 22 all cross-references but one are made to problems *in the same chapter*, not to problems in Chapters 14 and 15.[15] The one exception is in 22.6: "Do thus as I have also said to you above" (p. 352), referring to the rule to multiply the diameter of a circle by $3\frac{1}{7}$ to get the circumference. This rule is stated earlier in 15.2 (p. 284), but it is such a common rule in abbacus texts that the reference in 22.6 could easily be to another part of the (now lost?) book from which the chapter was taken. Or, equally likely, it may have been inserted by the scribe responsible

[14] In four independent instances **V** restates a problem in proportion as one in which the profits of a transaction are divided proportionally among business partners (pp. 12, 22). This occurs in two chapters found in **F** and **M**, as well as two that are not: 14.9 (which leads to its use in 14.10), 16.3, 21.4 (which leads to 21.8), and 22.1.

[15] 22.4 refers to a previous calculation in the same problem; 22.8, 22.22, and 22.32 refer to the previous problem; 22.21 refers to the previous two problems; 22.31 refers to the method of single false position, which is performed in previous problems in Chapter 22.

for collecting together the different parts of **V**. Last, contrary to H.'s assertion, the wide range of parameters chosen for mixed problems in abbacus texts make non-repetition likely.

Now I turn to evidence that the algebra in **V** does *not* belong to the 1307 original. Warren van Egmond provides the best argument in his article, "The study of higher-order equations in Italy before Pacioli" [2008]. Most abbacus algebras classify and solve some collection of equations of degree higher than 2.[16] The number of possible equations is quite large, especially because some books differentiate between terms like ax^2 and $\sqrt{ax^4}$. Dardi's 1344 tract classifies 194 types of equation, though the vast majority of books have fewer than fifty. To trace textual influences, van Egmond studied the selection and the order of the higher degree equations in 91 manuscripts, which he classified into families. Within each family there is a general trend over time toward increasing complexity and completeness. In the "Gherardi" family the 23 texts begin with Paolo Gherardi's list of 15 equations, and culminate in the late 15th c. with Raffaello Canacci's 63 equations. Van Egmond notes that Chapter 17 of **V** belongs to the Benedetto family. This family consists of eight texts, dating from ca. 1390 to ca. 1470. The earliest representative, titled *Tratato Sopra l'Arte Arismetricha* (*Treatise on the Art of Arithmetic*), gives the same list of equations as Chapter 17 of **V**, plus two more at the end. If Chapter 17 really dates to 1307, then the Benedetto family has a very large gap in time which cannot be accounted for. Thus Van Egmond writes that Chapter 17 "was undoubtedly added to a manuscript containing some sections copied from Jacopo's earlier work."[17]

The general trend in abbacus manuscripts from simple to complex, and toward improved organization, is turned upside down if we place **V**'s Chapter 17 in 1307. This chapter is more complete than any of the other early algebras (discounting Dardi, who is in a class by himself), and presents all but two reducible cubic and quartic equations in a very logical order, against the jumble we find in Gherardi and other texts from the first half of the 14th c. In fact, H.'s table on p. 160 would make much more sense if we move **V** *after* his **G** (Gherardi, 1328), **L**, and **C** (both from an anonymous manuscript, ca. 1330).

There is another abbacus book which contains Chapter 16 of **V**, covering equations of degrees 1 and 2. This chapter is found on ff. 28r-31v of a ca. 1365 *Trattato dell'Alcibra Amuchabile* (*Algebra Treatise*), which H. labels **A**.[18] The material preceding and following these pages are unrelated to anything in **V**. In particular, the treatment of higher degree equations in this book belongs to the Gherardi family. So while Chapter 17 of **V** belongs to a tradition which began later in the century, **V**'s Chapter 16 was originally written no later than 1365. That the two were not written together can also be seen by the fact that the earlier chapter contains no introduction or conclusion (like other 14th c. algebras), while Chapter 17 starts with, "Here I end the six rules combined with various examples. And begins the other rules that follow the six told above, as you will see." (p. 320) So the person who added Chapters 16 and 17 to the predecessor of **V** took the chapters from different books, or took them from a manuscript which had already combined them. There is no reason, then, to follow H.'s suggestion that Chapter 16 once began with a now lost introduction (pp. 101, 166 n. 320, 179).

Last, even H. acknowledges that Chapter 18, consisting of a single problem about grain, does not seem to fit with the surrounding chapters. The best explanation is that it is not Jacopo's, and was added along with the other chapters (pp. 100, 115).

16 Where solutions were given to irreducible cubic and quartic equations, they are false.
17 [Van Egmond 2008, 313].
18 Florence, Biblioteca Riccardiana Ricc. 2263 [Anonymous 1994].

6. Claim (1b): Jacopo's algebra was the very first, and derives from "area ?"

H. is not just pushing for Chapters 16 and 17 **V** to be the earliest known Italian treatment of algebra. He even maintains that it was the *very first* abbacus algebra ever written! "[Jacopo] was apparently the first to introduce the solution to the six fundamental cases and to (most of) those cases of the third and fourth degree that can be reduced by simple means... He also appears to have introduced the habit of applying algebra to *muʿāmalāt*-[i.e. merchant-]problems." (p. 181-2) The only evidence I can find for this assertion is on p. 153, where he notes that the algebra in **V** "avoids all abbreviations in the technical algebraic terminology, as if the author was conscious of introducing a new field of knowledge where readers would be unfamiliar with the terminology and therefore unable to expand abbreviations correctly." (see also pp. 9, 167) Other 14th c. algebras also have no abbreviations. So are we to presume that their authors were also conscious of "introducing a new field of knowledge"?[19]

H. continues to rely on his intuition rather than objective facts in his location of the source of Italian algebra. Recognizing that other 14th c. algebras do not depend *wholly* on Jacopo, H. assigns this extra influence to what he calls "area ?". Referring to a stemma he created from the false assumption that the algebra in **V** belongs to Jacopo, he writes:

> since **A**, **L** and **C** are all written in Tuscan with no traces of non-Tuscan orthography, even $\mathbf{A}''$ and $\mathbf{A}'$ [predecessors of **A**] are likely to have been written in Tuscan;[20] if this is so, then Gherardi must have sought his inspiration in Italian writings and found little of algebraic interest in Montpellier. But if there was no strong environment practising algebra in Montpellier in 1328, there can hardly have been any in 1307. (p. 167)

Casual guessing based on an incorrect stemma does not constitute a good argument. Later on, H. narrows down "area ?" as "indeed one area, to be identified with, located in or encompassing the Catalan region" (p. 181). This is "the only Romance-speaking area outside Italy where the next 150 years offer any evidence of algebraic interest" which does not include the area of Montpellier. But if we stick to the *fact* that both Jacopo and Paolo Gherardi wrote in Montpellier, then Montpellier remains the best candidate for the environment from which early abbacists drew their material, including algebra.

7. Claim (2): Abbacus books are not abridgements of Fibonacci's opus

In his chapter "The Abbacus Tradition" H. presents evidence that Fibonacci's *Liber Abaci* is not the primary source for early abbacus books. He compares the contents of Fibonacci's book with the earliest extant abbacus treatise, the late 13th c. *Livero de l'Abbecho* (*Abbacus Book*). The anonymous author of this book wrote that he compiled it "according to the opinion of master Leonardo Fibonacci" (p. 31), but H. shows that the contents betray the claim. In fact, only two of the 15 chapters of this book contain material taken from the *Liber Abaci*. Perhaps the half century separating the *Liber Abaci* from the *Livero de l'Abbecho* was enough for the Latin work to be condensed, reworked, and amended by an oral tradition, making the nod to Leonardo something more than just "an instance of embellishment" to a "culture hero" (p. 40).

[19] Arrighi writes of the ca. 1330 Lucca MS "Le equazioni, con le denominazioni di "cose", "censi", "chubj", "censi de li censi", sono ancora descritte non conoscendosi ancora una corrispondente tachigrafia." [Anonymous 1973, 11]. Likewise the algebra in Florence Fond. Prin. II.IX.57, part III (f. 171v), from ca. 1340, has no abbreviations.

[20] H. has "Tuscan area" here. I think the word "area" should be omitted.

H. then suggests that Fibonacci himself borrowed from an already-existing Italian environment. The evidence is slight, but intriguing. Fibonacci slips in a few Italian terms (p. 43), including *avere* for the Latin *census*.[21] Just how much of his book came from an Italian tradition we cannot know. It is certainly not the whole book, since we know that he borrowed from many written sources, including the algebras of al-Khwārizmī and Abū Kāmil.

8. Claim (2b): Abbacus algebra does not derive from Fibonacci or the Latin translations

For those who are not familiar with medieval algebra, special names were given to the powers of the unknown. The first power, our x, was called *shay'* in Arabic, *res* in Latin, and *cosa* in Italian. All these words mean "thing". The first power was also sometimes called "root": *jidhr* in Arabic, *radix* in Latin, and *radice* in Italian. The second power (x^2) was called *māl*, *census*,[22] and *censo*, all meaning "possession", "sum of money", or "treasure". The third power (x^3) was expressed by a word meaning "cube": *ka'b*, *cubus*, and *cubo*. Polynomials and equations were written out rhetorically, as will be seen in the quotes below.

H. sets aside a good portion of his book to the question of the sources of abbacus algebra. Here, too, he is correct that it does not derive from the known Latin books, nor from the Arabic originals from which translations were made. But his arguments are frequently completely wrong. Below I comment on the topics of "Normalization", "Examples", and "*Jabr* and *muqabala*" which he presents on pp. 155-158. H. has been repeating these errors in one article after another,[23] so I feel it is time to address them.

Under "Normalization" H. takes note of whether or not the six equations of degree 1 and 2 are presented in normalized form—that is, whether or not the coefficient of the highest power term is 1 (pp. 107, 148, 155). **V** and other abbacus texts state non-normalized equations. For example, in **V**'s fourth equation, "Quando li censi et le cose sonno oguali al numero" ("When the *censi* and the things are equal to the number", $ax^2 + bx = c$), the word *censi* is plural.

For Arabic/Latin algebra, H. writes, "Al-Khwārizmī's original text, like the Latin translations, defines all cases except 'things made equal to number' in normalized form and gives corresponding rules" (p. 155). Now the Oxford Arabic MS of al-Khwārizmī's *Algebra* and Robert of Chester's Latin translation both give the plural. In the translation of Gerard of Cremona, however, the generic type 6 equation has the dative singular *censui*. The other equations have the nominative *census*, a word which is both the singular and plural form. H. writes in a 2001 article, "we may be confident that Gerard's singular form corresponds to the original text."[24]

How can H. suggest that the reading of a single word in Gerard's translation is to be preferred over an Arabic MS and Robert's version? In his [1998] H. argues, based on mainly grammatical considerations, that Gerard of Cremona was a very conscientious translator, and that the 14th c. Oxford manuscript is corrupt.[25] It is true that there are several places where Gerard is closer to al-Khwārizmī's original, but let me point out one place where Gerard erred, and Oxford is correct. In the solution to problem (1) al-Khwārizmī has reached the equation "ten things less a *māl* equals twenty-one" ($10x - x^2 = 21$).

[21] This substitution occurs where the *census* (from the Arabic *māl*) takes the meaning of "quantity", not the algebraic second degree unknown. See my comments on Claim (2b) below.
[22] Robert of Chester uses *substantia* in place of *census*.
[23] See his [2000, 57-58; 2001, 23, 26, 27; 2001b, 113; 2006, 8-12, 15-17].
[24] [Høyrup 2001b, 113 n. 46].
[25] At the time the only two published editions of al-Khwārizmī's *Algebra* were based on this one manuscript.

The Arabic text then reads (literally) "So restore the ten the things..."[26] (i.e. "So restore the ten things"), while Gerard's "Restaura igitur decem excepta re"[27] ("Then restore ten less a thing") is mathematically incorrect.[28] However conscientious Gerard may have been, his translation is not immune from error.

When H.'s book went to press he had not yet seen Rashed's 2007 critical edition of al-Khwārizmī's *Algebra*.[29] We now know that all four surviving Arabic manuscrips have the plural in all equations. It makes much more sense that Gerard committed a blunder with his one word *censui* than to suppose that all Arabic manuscripts and Robert's translation are wrong. Besides, if al-Khwārizmī wished to present normalized equations, then why are two of the three sample equations for type 4 non-normalized, and why would he explain for types 5 and 6 that one must first perform the normalization? These examples are of course in Gerard's translation, too, contradicting the singular *censui* in equation 6.[30]

Getting back to his main argument, H. writes "In all the Latin treatises, all cases except "roots equal number"... are defined as normalized problems..." (p. 148) This is false. Robert of Chester's translation of al-Khwārizmī and the Latin translation of Abū Kāmil's *Algebra* both give the plural.[31] So even if we disregard his error about al-Khwārizmī's equations, there is nothing H. can conclude about normalization.

Under the heading "Examples", H. confuses the sample equations found in Arabic and Latin works with the sample worked-out problems (within which equations are set up) given after each rule in **V** (and indeed in most abbacus texts) (pp. 149-150, 155-6). He calls both sample equations and problems "examples": "The first observation to make concerning Jacopo's examples [read: worked-out problems] is that none of them are stated in terms of number, *things* and *censi* (afterwards, of course, a "position" is made identifying some magnitude with the *thing*, without this position, no reduction to the corresponding case sould result). In the Latin treatises, in contrast, the basic examples [i.e. equations] are always stated directly in the same number-roots-*census* terms as the rules." (pp. 149-50).

To clear up the confusion, compare al-Khwārizmī's rule for type 6 equations with the corresponding rule in **V**. Note that al-Khwārizmī explains the rule in the case of the specific equation $3x + 4 = x^2$, while **V** gives the general rule for $ax^2 = bx + c$:

> **al-Khwarizmī**:
>
> And for the roots and the number which equals the *māl*s. So suppose [someone] said to you: three roots and four in number equals a *māl*. So the way [to do this] is that you halve the roots, so it yields one and a half. So multiply it by its same, so it yields two and a fourth. So add it to the four, so it yields six and a fourth. So take its root, which is two and a half. So add it to half the roots, which is one and a half, so it yields four, which is the root of the *māl*, and the *māl* is sixteen. And if there is more than a *māl* or less, then return it to one *māl*.[32]
>
> **V**:

[26] [Al-Khwārizmī 2007, 157 line 11].
[27] [Hughes 1986, 250 line 6].
[28] Gerard's misreading is understandable, since the Arabic article *al-* ("the") looks identical to the word *illā* ("less") when placed before "thing" in unpointed text. But the case ending of the noun and the *al-* before "ten" should have tipped him off.
[29] [Al-Khwārizmī 2007].
[30] I have even more to say on this, but I should stop here.
[31] Fibonacci has the singular *censui* for the same equation as Gerard, the probable source of his error. He also gives instruction to normalize the equations.
[32] [Al-Khwārizmī 2007, 107].

> When the *censi* are equal to the things and to the number, one shall divide in the *censi*, and then halve the things, and multiply by itself and join to the number. And the root of the total plus the halving of the things is worth the thing. (p. 318).

Both books give worked-out problems for each type of equation, but they are organized differently. Al-Khwārizmī explains the solutions to the six equations and other preliminary topics first, and then gives six sample problems together, one for each type. **V**, by contrast, integrates the worked-out problems with the rules.

The sample type 6 problem in **V** is given immediately after the rule: "Example of the said rule. And I want to say thus, somebody has 40 gold *fiorini* and changed them into *venetiani.* And then from those *venetiani* he grasped [i.e. took] 60 and changed them back into gold *fiorini* at one *venetiano* more per *fiorino* than he changed them at first for me. And when he has changed thus, that one found that the *venetiani* which remained with him when he detracted 60, and the *fiorini* he got for the 60 *venetiani*, joined together made 100. I want to know how much was worth the *fiorino* in *venetiani.*" (pp. 319-20) In the solution the author posits "that the *fiorini* was worth one thing" and then works out the operations and sets up an equation which simplifies to "40 *censi* are equal to 120 things and 100 numbers" ($40x^2 = 120x + 100$). Then the rule is applied to get the answer.

Al-Khwārizmī's sample problem begins: "And the sixth problem. A *māl*: you multiplied its third by its fourth, so it brings back the *māl*, with twenty-four dirhams added".[33] In the solution he names the *māl* a "thing", then he works the operations to set up the type 6 equation "half a sixth of a *māl* equals a thing and twenty-four dirhams" ($\frac{1}{2}\frac{1}{6}x^2 = x + 24$). He then applies the rule to find the solution.

Note that al-Khwārizmī's enunciation is stated in terms of a *māl*, the same word used for the second power in algebra. The Arabic words *māl* ("sum of money", "possession", or simply "quantity") and *jidhr* ("[square] root") were appropriated as the names of algebraic powers from a common stock of arithmetic problems. In the enunciation to al-Khwārizmī's problem *māl* is a common noun meaning "quantity", while in the solution it is the name of the second power of the unknown. There is no confusion about the role of the term, since equations are always stated as the equality of two aggregations of the powers using the verb *ʿadala*, while the enunciations are usually expressed as a sequence of operations.[34]

While al-Khwārizmī's sample worked-out problems for types 4 and 6 are stated in terms of a *māl*, his other sample problems are "divided ten" questions. The enunciation to the third problem is "And the third problem. Ten: you divided it into two parts. Then you divided one of them by the other, so the result of the division is four."[35]

So when H. contrasts the "examples" in Arabic/Latin algebra, stated in terms of number, things, and *māls*/*census*, with **V**'s "examples" which are stated as abstract arithmetic or mercantile problems, he is really comparing simplified equations with worked-out problems, which are different kinds of objects. Once the distinction between equations and problems is recognized, the real differences between al-Khwārizmī's and **V**'s presentations become clear. First, the Arabic gives the rules in the context of specific equations while the Italian gives the general rules. Second, in Arabic the worked-out problems are grouped together after the rules, while in Italian they are integrated with the rules. Because H. is concerned with possible influence from Fibonacci, I note that the *Liber Abaci* gives the general rule (like **V**) followed by a sample equation (like al-Khwārizmī). So we cannot say that abbacus algebra was not influenced by Fibonacci on this point. But none of the Latin algebras integrate the worked-out

33 [Al-Khwārizmī 2007, 155].

34 There are many other indicators which I have no room to list here. On the different uses of *māl*, see [Oaks & Alkhateeb 2005]. This article, and [Oaks 2009], address the distinction between equations and enunciations.

35 [Al-Khwārizmī 2007, 149].

problems with the rules, so in this respect abbacus algebra was not influenced by Fibonacci or the Latin translations.

Last is "*Jabr* and *muqabala*". Here H. relies on some scattered translations and Saliba's outdated article to give a short and confused account of the uses of *al-jabr* ("restoration") and *al-muqābala* ("confrontation") in Arabic algebra (pp. 105, 151-2, 157-8). For the uses of these and other terms, see instead my article "Simplifying equations in Arabic algebra" [Oaks & Alkhateeb 2007]. H. is correct that the use of *ristorare* in **V** differs from the use of *al-jabr* in Arabic algebra, but he does not adequately explain how either term is used, nor does he situate *ristorare* in the context of other abbacus texts. I will do this briefly now.

The two basic steps in the simplification of a polynomial equation in Arabic algebra are to "restore" (*al-jabr*) diminished terms and to "confront" (*al-muqābala*) like terms on opposite sides of the equation. An example of *al-jabr* is converting $10x - x^2 = 21$ to $10x = 21 + x^2$. In this step al-Khwārizmī writes "So restore the ten things [$10x$] by the *māl* [x^2], and add it to the twenty-one…".[36] The $10x - x^2$ was considered to be a diminished $10x$ with no operation present, so one must "restore" it to a full $10x$. One then must add the x^2 to the other side to balance the equation. An example of *al-muqābala* takes us from $3\frac{1}{3} + \frac{1}{3}x = x$ to $3\frac{1}{3} = \frac{2}{3}x$. Like terms are "confronted", resulting in their difference on one side. In Arabic the phrase *al-jabr wa'l-muqābala* ("restoration and confrontation") was used as a shortcut to mean "perform the necessary restoration and/or confrontation to simplify the equation".[37] It is important to keep in mind that *al-jabr* and *al-muqābala* were not technical terms. They were employed in Arabic algebra for these as well as other purposes following their ordinary meanings.

In abbacus algebra two different phrases (with the usual orthographic variations) can be found with the same meaning as *al-jabr wa'l-muqābala*: *ristora le parti* ("restore the parts") and *uguagliare le parti* ("equalize the parts").[38] "The parts" (*le parti*) are the two sides of the equation. For example, **V** has "120 cose meno 40 censi et più 160 numeri sonno oguali a 60. Ristora ciascheuna parte, arai che 40 censi sonno oguali a 120 cose et 100 numeri." ("120 things less 40 *censi* and added 160 numbers are equal to 60. Restore each part, you will get that 40 *censi* are equal to 120 things and 100 numbers." (p. 319)) Here both a restoration and a confrontation are performed, as we also see on p. 317. In three other examples in **V** the phrase entails only a confrontation, and in one only a restoration.

Among the ca. 30 abbacus algebras I have read, only two others use *ristora le parti*. These are an anonymous piece from ca. 1330 (H.'s **L**, Lucca MS 1754) and Piero della Francesca's ca. 1480 *Trattato d'Abaco* (*Abbacus Treatise*).[39] Many more books apply the phrase *uguagliare le parti*. Here is an example from Maestro Biagio, in a redaction of Maestro Benedetto (1463): "4 censj e questi sono igualj a 25 meno uno censo; dove raguaglieraj le partj, dando a ogni parte 1 censo e aremo 5 censj igualj a 25"[40] ("4 *censi*, and these are equal to 25 less a *censo*; where you will equalize the parts, giving to each part 1 *censo*, and we will have 5 *censj* equal to 25").[41]

H.'s other evidence for the independence of abbacus algebra on pages 154-158 is correct, though he is wrong that al-Khwārizmī "clearly considers the *root* as the unknown proper" (and not the *māl*, pp. 102, 156). The unknown number in the enunciation to a problem was usually named "thing" (i.e.

36 [Al-Khwārizmī 2007, 157 line 11].
37 This phrase is also the name of the art of algebra in Arabic.
38 I have found only one example of a text which includes other operations. Filippo Calandri (ca. 1495) is inconsistent, sometimes using *raguagla le parti* properly, and other times including operations like multiplying through by a denominator of one *cosa* or setting the highest power term to 1. [Calandri 1982, 6, 18, 31, 35].
39 [Anonymous 1973; Piero della Francesca 1970]. **A** of course uses the same phrase as **V**, since it contains the same text.
40 [Biagio 1983, 52-53].
41 I have found one book which sometimes has the phrase *diffa e debiti* for the same purpose [Canacci 1983].

"root"), but very often it was named *māl*. The "unknown proper" is whatever algebraic power (or even expression) you assigned your unknown in the beginning of the solution.[42]

9. Concluding remarks

Many readers of this review will wonder how such a prominent publication can contain so many fundamental mistakes. After all, H.'s claims have been published before in good journals. It turns out that his first paper on Jacopo ([2001]), containing the details of his argument that the algebra dates to 1307, never passed through a peer review. It is part of a 1999 conference proceedings which was evidently distributed only to the participants. In his *Centaurus* article of 2000 he writes "I intend to publish this evidence elsewhere",[43] and in his 2006 *Historia Mathematica* article he refers to the 2001 "publication" and says "Repetition of the details of the extensive argument would lead too far".[44] It is likely that the few peers who saw H.'s [2001] did not put in the effort to evaluate it critically.

Despite H.'s efforts to show that Jacopo was first, Paolo Gherardi's 1328 *Libro di Ragioni* remains the oldest known Italian book with algebra, and he probably learned this algebra in the environment of Montpellier. On the other hand, H. has done the field a great service by pointing out that abbacus texts are not merely vulgarized compendia of Fibonacci's *Liber Abaci*, a finding which should encourage further studies in abbacus mathematics.

Keeping in mind all its faults, H.'s book has a lot to offer about abbacus texts and their place in the broader history of mathematics. The best part is the edition and translation of **V**, which will give even the casual reader a taste of mathematics of a different kind than we learn today, one written and cultivated by trade groups for pedagogical and practical reasons, rather than by theoretical mathematicians who pursue knowledge for its own sake.

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[42] See [Oaks & Alkhateeb 2005, 408].
[43] [Høyrup 2000, 65 n. 9].
[44] [Høyrup 2006, 7 n. 5].

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